Exam sets October 2024

Always explain your answers. It is allowed to refer to definitions, lemmas and theorems from the lecture notes but not to other sources. All questions are independent and count equally so make sure you try each of them. Good luck!

0. Define the function $f : \mathbb{N} \to \mathbb{Q}$ by $f(x) = 6$ for all $x \in \mathbb{N}$. Describe the set $f^{-1}(\{0, 1\})$ explicitly and explain your answer.

1. Give an explicit example of a set X and an injective function $f : X \to X$ that is not surjective. Make sure you prove your statements.

2. **Explain what is wrong the following reasoning:** For all $n \geq 10^{2024}$ we have $2^n < 2^{n-1}$. This is proven by induction as follows: Define for every $n \in \mathbb{N}$ the statement S_n to be $2^{2024+n} < 2^{2023+n}$. Now suppose S_n holds for some $n \in \mathbb{N}$ then we will show S_{n+1} must also be true. To prove S_{n+1} we multiply both sides of the inequality $2^{2024+n} < 2^{2023+n}$ (guaranteed by S_n) by 2 to obtain $2^{2024+n+1} < 2^{2023+n}$. This proves that S_n implies S_{n+1} for all n. By induction we have thus demonstrated that S_n holds for all $n \in \mathbb{N}$.

3. Imagine a set A with at least two elements. Define a relation $\sim$ on $A \times A$ by setting $(a, b) \sim (c, d)$ iff $a = d$ and $b = c$. Is $\sim$ an equivalence relation on $A \times A$?

4. Recall that $[k] = \{0, 1, \ldots k - 1\}$. Find the cardinality of $\bigcap_{n \in \mathbb{N}} [n + 1]$ and prove your result.

5. Prove that if A, B and C are sets then $(A \setminus B) \times C \subseteq A \times C$.

6. Define $F : \mathbb{Z}^{\mathbb{Z}} \to \mathbb{Z}$ by $F(g) = (g \circ g \circ g)(0)$ for any $g \in \mathbb{Z}^{\mathbb{Z}}$. Find the range of F and prove your claim.

7. Suppose we have an equivalence relation $\sim$ on $\{a, b, c, d\}$ and assume that $a \sim b$ and $c \sim b$. Show that $\{a, b, c\}$ is a subset of the equivalence class of c.

8. Define a sequence of subsets $A_0, A_1, A_2, \ldots$ of $\mathbb{N}$ as follows. $A_0 = [0]$ and $A_1 = [1]$ and for $n \geq 2$ we set $A_n = A_{n-1} \cap A^c_{n-2}$, where $X^c = \mathbb{N} \setminus X$. Prove that for all $n \in \mathbb{N}$ the set A_n is finite.

9. Prove that if sets A and B are both uncountable sets then $A \cup B$ is uncountable as well.

Solutions

0. Define the function $f : \mathbb{N} \to \mathbb{Q}$ by $f(x) = 6$ for all $x \in \mathbb{N}$. Describe the set $f^{-1}(\{0, 1\})$ explicitly and explain your answer.
   $f^{-1}(\{0, 1\}) = \{n \in \mathbb{N} : f(n) \in \{0, 1\}\} = \emptyset$ because the function f never takes the values 0 or 1.☺

1. Give an explicit example of a set X and an injective function $f : X \to X$ that is not surjective. Make sure you prove your statements.
   Take $X = \mathbb{N}$ and define f by $f(n) = n + 1$. Then f is injective because if $f(x) = f(y)$ for some $x, y \in \ltimes$ then $x + 1 = y + 1$ so $x = y$. However f is not surjective since $f(x) > 0$ holds for all n since adding one to a natural number produces a positive number. In other words 0 is not in the range of f as required☺

2. **Explain what is wrong the following reasoning:** For all $n \geq 10^{2024}$ we have $2^n < 2^{n-1}$. This is proven by induction as follows: Define for every $n \in \mathbb{N}$ the statement S_n to be $2^{2024+n} < 2^{2023+n}$. Now suppose S_n holds for some $n \in \mathbb{N}$ then we will show S_{n+1} must also be true. To prove S_{n+1} we multiply both sides of the inequality $2^{2024+n} < 2^{2023+n}$ (guaranteed by S_n) by 2 to obtain $2^{2024+n+1} < 2^{2023+n}$. This proves that S_n implies S_{n+1} for all n. By induction we have thus demonstrated that S_n holds for all $n \in \mathbb{N}$.
   The induction basis, the case S_0 was never checked and is indeed false: $2^{2024} = 2 * 2^{2023} > 2^{2023}$.

3. Imagine a set A with at least two elements. Define a relation $\sim$ on $A \times A$ by setting $(a, b) \sim (c, d)$ iff $a = d$ and $b = c$. Is $\sim$ an equivalence relation on $A \times A$?
   No because it is not reflexive: for example if x, y are two distinct elements in A then (x, y) is not equivalent to x, y because $x \neq y$.

4. Recall that $[k] = \{0, 1, \ldots k - 1\}$. Find the cardinality of $\bigcap_{n \in \mathbb{N}} [n + 1]$ and prove your result.
   The intersection contains those elements contained in all the sets $[n + 1]$ where n runs throught the natural numbers but the case $n = 0$ yields a set with a single element $[1] = \{0\}$. All other sets $[n + 1]$ also contain 0 by definition so the intersection is $\{0\}$ and its cardinality is 1 because it contains one element and $\mathrm{id}_{[1]}$ provides an invertible function demonstrating the cardinality ☺

5. Prove that if A, B and C are sets then $(A \setminus B) \times C \subseteq A \times C$.
   Any element $x \in (A \setminus B) \times C$ must be of the form $x = (r, c)$ where $r \in A \setminus B$ and $c \in C$. In particular $x \in A \times C$ because $x = (r, c)$ and we already stated that $r \in A$ and $c \in C$ as required ☺

6. Define $F : \mathbb{Z}^{\mathbb{Z}} \to \mathbb{Z}$ by $F(g) = (g \circ g \circ g)(0)$ for any $g \in \mathbb{Z}^{\mathbb{Z}}$. Find the range of F and prove your claim.

For any $z \in \mathbb{Z}$ consider the constant functions $C_z \in \mathbb{Z}^{\mathbb{Z}}$ defined by $C_z(n) = z$. Then $C_z \circ C_z = C_z$ because for any n we have $(C_z \circ C_z)(n) = C_z(C_z(n)) = C_z(z) = z = C_z(n)$ and hence also $C_z \circ C_z \circ C_z = C_z$. This means that $F(C_z) = (C_z \circ C_z \circ C_z)(0) = C_z(0) = z$. It allows us to prove that F is surjective and so the range is $\mathbb{Z}$ because for any $z \in \mathbb{Z}$ we have $F(C_z) = z$ as shown above.

7. Suppose we have an equivalence relation $\sim$ on $\{a, b, c, d\}$ and assume that $a \sim b$ and $c \sim b$. Show that $\{a, b, c\}$ is a subset of the equivalence class of c.
   By symmetry we have $b \sim c$ and by transitivity we thus get $a \sim c$ showing that $a, b, c \in \bar{c}$ meaning $\{a, b, c\} \subseteq \bar{c}$ as required ☺

8. Define a sequence of subsets $A_0, A_1, A_2, \dots$ of $\mathbb{N}$ as follows. $A_0 = [0]$ and $A_1 = [1]$ and for $n \geq 2$ we set $A_n = A_{n-1} \cap A_{n-2}^c$, where $X^c = \mathbb{N} \setminus X$. Prove that for all $n \in \mathbb{N}$ the set A_n is finite.
   We use induction to prove that for every $n \in \mathbb{N}$ the statement S_n that says: A_n is finite. The induction basis is the case S_0 and by definition $A_0 = [0]$ is a finite set. Now assume that S_n holds for some n (this is our induction hypothesis). Then we will demonstrate that it follows that S_{n+1} is also true. In case $n = 0$ we already know that $S_{n+1} = S_1 = [1]$ is finite. For $n > 0$ we write $A_{n+1} = A_n \cap A_{n-1}^c \subseteq A_n$. By S_n the set A_n is finite and we know from the lecture notes that a subset of a finite set is finite so A_{n+1} must be finite proving S_{n+1}. This finishes the induction proof and shows that S_n holds for all $n \in \mathbb{N}$. In other words A_n is finite for all natural numbers n ☺

9. Prove that if sets A and B are both uncountable sets then $A \cup B$ is uncountable as well.
   We give a proof by contradiction so assume that $X = A \cup B$ is countable. The idea is that if $f : X \to Y$ is invertible then the restriction $r : A \to f(A)$ is also invertible with inverse given by $r^{-1}(y) = f^{-1}(y)$.

   The first case to consider is X is finite, so $Y = [n]$ for some $n \in \mathbb{N}$. This implies that $f(A) \subseteq [n]$ is also finite since in Chapter 2 we proved that any subset of a finite set is finite. Using the invertible r defined above we find that A must also be finite which is absurd.

   The second case to consider is that $Y = \mathbb{N}$. To apply the same argument we need to demonstrate $W = f(A)$ is countable. To do so will give an invertible function $g : W \to \mathbb{N}$ below. The composition $g \circ f$ will then show A must also be countable.

   To define g first introduce for any $m \in \mathbb{N}$ the finite set $G_m = \{w \in W : w < m\}$. It is indeed finite since $G_m \subseteq [m]$ (see above). Now define $g(m) = \#G_m$. To show g is invertible we prove g is injective and surjective. Assume $v, w \in W$ are such that $g(v) = g(w)$ then we can assume that $w \leq v$ so that $G_w \subseteq G_v$. If $w < v$ then $w \in G_v$ but $w \notin G_w$ showing that $g(w) < g(v)$ so we conclude $w = v$. To show g is surjective we use induction to prove the statements S_n saying that there exists an $x \in W$ such that $g(x) = n$. As a basis we demonstrate S_0 by noting that $g(\min B) = 0$. Assuming S_n we have some x such that $\#G_x = n$ and consider the set $C = B \setminus G_x$. If $C\emptyset$ then $f(A) = B \subseteq G_x$ would be finite so A is a finite set. Therefore we can take the minimal element $c = \min C$ and note that $G_x \cup \{c\} = \{G_c\}$ and since $c \notin G_x$ we have $\#G_c = n + 1$ proving S_{n+1}. We thus found an invertible g as required ☺